COEFFICIENT BOUNDS FOR CERTAIN NEW SUBCLASS OF m-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH CONIC DOMAINS

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Abstract: Considering a function $f(z)$ which is analytic in the open unit disk E, normalized by $f(0)-0=f'(0)-1$ and having the power series of the form $f(z)=z+a_2z^2+a_3z^3+\ldots$. We introduce and study certain new subclass $M^{\mu,k}_{\Sigma,m}(\beta,\lambda,A,B)$ of bi-univalent function associated with conic domains in which both f and f' are m-fold symmetric analytic functions. It is pertinent to remark here that the first two initial bi-univalent coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ estimated in the present investigation have their application in the construction of some (and not limited to) analytic functional such as Fekete-Szego functional and Hankel determinants to mention but few.

Keywords: Coefficient bounds, bi-univalent functions, conic domains

Introduction

Let A denote the class of all functions having the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (1)$$

which are analytic in the open unit disk $E=\{z:z\in C,|z|<1\}$. Also, let S denote the class of all normalized analytic functions that are univalent in E. We shall recall that every function $f\in S$ has an inverse f^{-1}, given by;

$$f^{-1}(f(z))=z \quad (z\in E) \qquad (2)$$

and

$$f(f^{-1}(\omega))=\omega \qquad \left(|\omega|<\gamma_0(f);\gamma_0(f)\geq\tfrac{1}{4}\right), \qquad (3)$$

while the inverse function $g(\omega)=f^{-1}(\omega)$ is given by

$$g(\omega)=f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\ldots \qquad (4)$$

A function is said to be bi-univalent in E if both f and its inverse (i.e. f and f^{-1}) are univalent in E. The class of bi-univalent functions denoted by Σ has been widely studied by different authors from different perspective and their results authenticated diversely in literature (Brannan and Taha, 1986; Lewin, 1967) for more details. In particular, the studies of the bi-univalent function class Σ by Srivastava *et al*. (2010) has triggered the interest of several researchers in the recent years.

Suppose that $m\in N$. A domain D is said to be $m-$ fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{m}$ carries D on itself, then, a function $f(z)$ analytic in E is said to be $m-$fold symmetric if

$$f\left(e^{2\pi i/m}z\right)=e^{2\pi i/m}f(z)\ (m\in N). \qquad (5)$$

Here, it is pertinent to remark that every function $f(z)$ is one – fold (1- fold) symmetric and every odd function $f(z)$ is two - fold (2- fold) symmetric. Let us denote by S_m the class of $m-$fold symmetric univalent function in E. Then a trivial argument has shown that $f\in S_m$ has a power series of the form

$$f(z)=z+\sum_{p=1}^{\infty}a_{mp+1}z^{mp+1}\ (z\in E,m\in N). \qquad (6)$$

In particular, for the normalized form of f given by equation (6) above, Srivastava *et al*. (2014) defined $m-$ fold symmetric bi- univalent functions, analogues to the concept of $m-$fold symmetric univalent functions and obtained the series expansion for f^{-1} as;

$$g(\omega)=f^{-1}(\omega)=\omega-a_{m+1}\omega^{m+1}+[(m+1)a_{m+1}^2-a_{2m+1}]\omega^{2m+1}-\left[\tfrac{1}{2}(m+1)(3m+2)a_{m+1}^3-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]\omega^{3m+1}+\ldots\ldots \qquad (7)$$

Denoted by Σ_m, the class of $m-$ fold symmetric bi – univalent functions in E and for $m=1$, the expression in (7) immediately coincides with that of (4) of the class Σ (Bulut, 2016).

Although investigation on the class of $m-$ fold symmetric bi – univalent function (Σ_m) in E is not so new. The target of the present paper is to introduce and study a new subclass of bi – univalent functions in which both f and f^{-1} are $m-$ fold symmetric analytic functions associated with conic domain where the first two initial coefficients of the Taylor – Maclaurin series $|a_{m+1}|$ and $|a_{2m+1}|$ are estimated. However, Noor and Malik (2011) introduced and studied the classes *k*-starlike functions $k-ST[A,B]$ and k-uniformly convex functions, $k-UCV[A,B]$, using the following definitions:

A function $f(z)\in A$ is said to be in the class $k-ST[A,B]$, $k\geq 0$, $-1\leq B<A\leq 1$, if and only if;

$$\mathrm{Re}\left(\frac{(B-1)\frac{zf(z)}{f(z)}-(A-1)}{(B+1)\frac{zf(z)}{f(z)}-(A+1)}\right)>\left|\frac{(B-1)\frac{zf(z)}{f(z)}-(A-1)}{(B+1)\frac{zf(z)}{f(z)}-(A+1)}-1\right| \qquad (8)$$

Also, a function $f(z)\in A$ is said to be in the class $k-UCV[A,B]$, $k\geq 0$, $-1\leq B<A\leq 1$, if and only if

$$\mathrm{Re}\left(\frac{(B-1)\frac{(zf(z))'}{f(z)}-(A-1)}{(B+1)\frac{(zf(z))'}{f(z)}-(A+1)}\right)>\left|\frac{(B-1)\frac{(zf(z))'}{f(z)}-(A-1)}{(B+1)\frac{(zf(z))'}{f(z)}-(A+1)}-1\right| \qquad (9)$$

Obviously,

$$f(z)\in K-UCV[A,B]\Leftrightarrow zf'(z)\in K-ST[A,B]. \qquad (10)$$

Remark

(i) $k-ST[1,-1]=k-ST$, $k-UCV[1,-1]=k-UCV$, the well – known classes of $k-$starlike and $k-$ Uniformly convex functions respectively (Kanas and Wisniowska, 1999; 2000).

(ii) $k-ST[1-2\alpha,-1]=SD(k,\alpha)$, $k-UCV[1-2\alpha,-1]=KD(k,\alpha)$, (Shams *et al*., 2004)

(iii) $0-ST[A,B]=S^{*}[A,B]$, $0-UCV[A,B]=C[A,B]$, the well-known classes of Janowski starlike and Janowski convex functions, respectively (Janowski, 1973).

Geometrically, if a function $f(z)\in k-ST(A,B)$ then $\dfrac{(B-1)\frac{zf(z)}{f(z)}-(A-1)}{(B+1)\frac{zf(z)}{f(z)}-(A+1)}=\omega$ takes all values from the domain Ω_k, $k\geq 0$, as

$$\Omega_k=\{\omega: R(\omega)>k|\omega-1|\}. \qquad (11)$$

Or equivalently,

$$\Omega_k=\left\{u+iv; u>k\sqrt{(u-1)^2+v^2}\right\}. \qquad (12)$$

It is worthy to note that the domain Ω_k represents the right half plane for $k=0$, a hyperbola for $0<k<1$, a parabola for $k=1$ and an ellipse for $k>1$ (Yagmur, 2015).

Now, for $\lambda\geq 1,\mu\geq 0,k\geq 0$, $m\in N$, $-1\leq B<A\leq 1, 0\leq\beta<1$ and $z,\omega\in E$, a function $f\in\Sigma_m$ given by (6) is said to be in the class $M^{\mu,k}_{\Sigma,m}(\beta,\lambda,A,B)$ if and only if;

$$\mathrm{Re}\left\{\frac{(B-1)\left(\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]-\beta\right)-(A-1)}{(B+1)\left(\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]-\beta\right)-(A+1)}\right\}$$
$$>k\left|\frac{(B-1)\left(\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]-\beta\right)-(A-1)}{(B+1)\left(\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]-\beta\right)-(A+1)}-1\right| \qquad (13)$$

and

$$\mathrm{Re}\left\{\frac{(B-1)\left(\left[(1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}\right]-\beta\right)-(A-1)}{(B+1)\left(\left[(1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{f(\omega)}{\omega}\right)^{\mu-1}\right]-\beta\right)-(A+1)}\right\}$$
$$>k\left|\frac{(B-1)\left(\left[(1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}\right]-\beta\right)-(A-1)}{(B+1)\left(\left[(1-\lambda)\left(\frac{g(\omega)}{g}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}\right]-\beta\right)-(A+1)}-1\right|. \qquad (14)$$

Remark

For suitable choices of the parameters μ, λ, m, A, B and k. The class $M_{\Sigma,m}^{\mu,k}(\beta,\lambda,A,B)$ leads to certain interesting classes (known and new) of analytic bi-univalent functions. As illustrative example, we give the following:

Illustration 1

For $A = m = 1$, $k = 0$ and $B = -1$, we have the bi – univalent function class

$M_{\Sigma,1}^{\mu,0}(\beta,\lambda,1-1) = M_{\Sigma}^{\mu}(\beta,\lambda)$

introduced by Caglar *et al.* (2013).

Illustration 2

For $A = \lambda = 1$, $k = 0$ and $B = -1$, we have a new class

$M_{\Sigma,m}^{\mu,0}(\beta,1,1,-1) = P_{\Sigma,m}(\beta,\mu)$

which consist of $m-$ fold symmetric bi – Bazilevic functions.

Illustration 3

For $A = \lambda = 1$, $k = \mu = 0$ and $B = -1$, we have the class

$M_{\Sigma,m}^{0,0}(\beta,1,1,-1) = M_{\Sigma,m}^{0}(\beta,1)$

of $m-$ fold symmetric bi – starlike functions of order β (Hamidi and Janjangiri, 2014).

Illustration 4

For $A = m = \lambda = 1$, $k = \mu = 0$ and $B = -1$, we have the bi – Starlike function class $M_{\Sigma,m}^{0,0}(\beta,1,1,-1) = S_{\Sigma}^{*}(\beta)$ introduced and studied by Brannan and Taha (1986).

Illustration 5

If $A = \mu = \lambda = 1$, $k = 0$ and $B = -1$, then we have the $m-$ fold symmetric bi – univalent function class $M_{\Sigma,m}^{1,0}(\beta,1,1,-1) = \mathrm{H}_{\Sigma,m}(\beta)$ introduced and studied by Srivastava *et al*. (2014).

Illustration 6

Suppose that $A = m = \mu = \lambda = 1$, $k = 0$ and $B = -1$, then we have the bi – univalent function class $M_{\Sigma,1}^{1,0}(\beta,1,1,-1) = H_{\Sigma}(\beta)$ introduced and studied by Srivastava *et al*. (2010).

Illustration 7

Let $A = \mu = 1$, $k = 0$ and $B = -1$, then we have the $m-$ fold symmetric bi – univalent function class $M_{\Sigma,m}^{1,0}(\beta,\lambda,1,-1) = \mathrm{A}_{\Sigma,m}^{\lambda}(\beta)$ introduced and studied by Sumer Eker (2016).

Illustration 8

Let $A = \mu = m = 1$, $k = 0$ and $B = -1$ then, we have the bi – univalent function class $M_{\Sigma,1}^{1,0}(\beta,\lambda,1,-1) = \mathrm{B}_{\Sigma}(\beta,\lambda)$ introduced and studied by Frasin and Aouf (2011).

Illustration 9

If $A = 1$, $k = 0$ and $B = -1$, then we have the class $M_{\Sigma,m}^{\mu,0}(\beta,\lambda,1,-1) = N_{\Sigma,m}^{\mu}(\beta,\lambda)$ which is the class of $m-$fold symmetric analytic bi-univalent function introduced and studied by Bulut (2016).

Illustration 10

For $m = 1$, then we have a new class $M_{\Sigma,1}^{\mu,k}(\beta,\lambda,A,B)$ which is the class of bi-univalent functions associated with conic domain.

Illustration 11

For $\lambda = 1$, then we have a new class $M_{\Sigma,m}^{\mu,k}(\beta,1,A,B)$ which is the class of $m-$fold symmetric bi-Bazelivic functions associated with conic domain.

Illustration 12

For $\mu = 0$ and $\lambda = 1$, we obtain a new class $M_{\Sigma,m}^{0,k}(\beta,1,A,B)$ of $m-$fold symmetric bi-starlike functions of order β associated with conic domains.

Coefficient estimates for the function class $M_{\Sigma,m}^{\mu,k}(\beta,\lambda,A,B)$.

Lemma 2.1 If $p \in P$, then $|c_n| \le 2$ $(n \in N)$, where the Caratheodory class P is the family of all functions p analytic in E for which $\mathbf{Re}[p(z)] > 0$, and

$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in E)$ (Bulut, 2016).

Theorem 2.2 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{\mu,k}(\beta,\lambda,A,B)$. Then, for $\mu \ge 0,\ \lambda \ge 1,\ -1 \le B < A \le 1,\ k \ge 0,\ 0 \le \beta < 1,\ m \in N$, $z \in E$

$$(\mu + m\lambda)a_{m+1} = -\frac{\delta(\alpha + 2Bk)p_m}{(1+k)[\alpha(B+1) - \delta(B-1)]} \tag{15}$$

$$\left[\delta(\mu-1)(\mu+2m\lambda) - 2(B+1)(\mu+m\lambda)^2\right]a_{m+1}^2 + 2\delta(\mu+2m\lambda)a_{2m+1} = -\frac{2\delta^2(\alpha+2Bk)p_{2m}}{(1+k)[\alpha(B+1)-\delta(B-1)]} \tag{16}$$

$$(\mu + m\lambda)a_{m+1} = \frac{\delta(\alpha + 2Bk)q_m}{(1+k)[\alpha(B+1) - \delta(B-1)]} \tag{17}$$

$$\left[(B+1)(\mu+m\lambda)^2-\delta(\mu+2m\lambda)\left(m+\frac{\mu+1}{2}\right)\right]a_{m+1}^2+\delta(\mu+2m\lambda)a_{2m+1}$$
$$=\frac{\delta^2(\alpha+2Bk)q_{2m}}{(1+k)[\alpha(B+1)-\delta(B-1)]} \tag{18}$$

where $\alpha=(B-A)-\beta(B-1)$, $\delta=(B-A)-\beta(B+1)$ and $\mu,\lambda,A,B,\beta,m,k$ are as earlier defined.

Proof: From (13), setting

$$\frac{(1+k)\left\{\dfrac{(B-1)\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-\beta\right)-(A-1)}{(B+1)\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-\beta\right)-(A+1)}\right\}-k}{\dfrac{(1+k)[(B-A)-\beta(B-1)]}{[(B-A)-\beta(B-1)\}}-k}=p(z).$$

Then, it follows that

$$(1+k)\left\{\frac{(B-1)\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-\beta\right)-(A-1)}{(B+1)\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-\beta\right)-(A+1)}\right\}-k=\left\{\frac{(1+k)[(B-A)-\beta(B-1)]}{[(B-A)-\beta(B+1)\}}-k\right\}p(z). \tag{19}$$

Likewise, from (14), setting

$$\frac{(1+k)\left\{\dfrac{(B-1)\left((1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}-\beta\right)-(A-1)}{(B+1)\left((1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}-\beta\right)-(A+1)}\right\}-k}{\dfrac{(1+k)[(B-A)-\beta(B-1)]}{(B-A)-\beta(B+1)}-k}=q(\omega).$$

Then, it follows that

$$(1+k)\left\{\frac{(B-1)\left((1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}-\beta\right)-(A-1)}{(B+1)\left((1-\lambda)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\lambda g'(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\mu-1}-\beta\right)-(A+1)}\right\}-k=\left[\frac{(1+k)[(B-A)-\beta(B-1)]}{[(B-A)-\beta(B+1)]}-k\right]q(\omega) \tag{20}$$

Where *p(z)* and *q(ω)* are defined as follow:

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\ldots\in P \tag{21}$$

and

$$q(z)=1+q_m\omega^m+q_{2m}\omega^{2m}+q_{3m}\omega^{3m}+\ldots\in P. \tag{22}$$

On equating the coefficients in (19) and (20), then

$$-(1+k)(\mu+m\lambda)\big[(B+1)\alpha-k\delta\big]a_{m+1}=\delta\big[(1+k)\alpha-k\delta\big]p_m,$$

$$\left[\delta(\mu-1)(\mu+2m\lambda)-2(B+1)(\mu+n\lambda)^2\right]a_{m+1}^2+2\delta(\mu+2m\lambda)a_{2m+1}=\frac{-2\delta^2(\alpha+2Bk)p_{2m}}{(1+k)[\alpha(B+1)-\delta(B-1)]},$$

$$(\mu+m\lambda)(1+k)\big[(B+1)\alpha-(B-1)\delta\big]a_{m+1}=\big[\alpha(1+k)-\delta k\big]\delta q_m$$

and

$$\left[(B+1)(\mu+m\lambda)^2-\delta(\mu+2m\lambda)\left(m+\frac{\mu+1}{2}\right)\right]a_{m+1}^2+\delta(\mu+2m\lambda)a_{2m+1}=\frac{\delta^2(\alpha+2Bk)q_{2m}}{(1+k)[\alpha(B+1)-\delta(B-1)]}. \tag{26}$$

where $\alpha=(B-A)-\beta(B-1)$ and $\delta=(B-A)-\beta(B+1)$. This completes the proof of Theorem 2.2.

Theorem 2.3 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{\mu,k}(\beta,\lambda,A,B)$. Then, for $\mu \ge 0,\ \lambda \ge 1,\ -1 \le B < A \le 1,\ k \ge 0,\ 0 \le \beta < 1,\ m \in N$, $z \in E$

$$|a_{m+1}| \le \sqrt{\frac{8\alpha\delta^2(\alpha+2Bk)}{\alpha(1+k)[R+T]}} \tag{23}$$

and

$$|a_{2m+1}| \le \frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{\delta(m+1)(\alpha+2Bk)}{(1+k)(\mu+m\lambda)^2[\delta(B-1)-\alpha(B+1)]}+\frac{1}{(\mu+2m\lambda)}\right] \tag{24}$$

where

$\alpha=(B-A)-\beta(B-1)$, $\delta=(B-A)-\beta(B+1)$, $R=2[\alpha(B+1)(1+\delta)-2\delta(B-1)](B+1)(\mu+m\lambda)^2$

and $T=\delta(\mu+2m\lambda)\left[\delta(\mu-1)(B-1-\alpha(B+1))-2(\alpha(B+1)-\delta(B-1))\left(m+\frac{\mu+1}{2}\right)\right]$.

Proof: It follows from (15) and (17) that

$$-p_m = q_m \tag{25}$$

and

$$(\mu+m\lambda)^2 a_{m+1}^2 = \frac{\delta^2(\alpha+2Bk)^2(p_m^2+q_m^2)}{2(1+k)^2[\alpha(B+1)-\delta(B-1)]^2}. \tag{26}$$

Now by the application of Lemma 2.1 on (26), we obtain the inequality in (23).

Upon subtracting (18) from (16) and substituting the value of a_{m+1}^2 in (26) we obtain the bound on the coefficient $|a_{2m+1}|$, as follows

$$a_{2m+1} = -\frac{\delta(\alpha+2Bk)}{2(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{\delta(m+1)(\alpha+2Bk)(q_m^2+p_m^2)}{2(1+k)(\mu+m\lambda)^2[\delta(B-1)-\alpha(B+1)]}+\frac{q_{2m}-p_{2m}}{(\mu+2m\lambda)}\right] \tag{27}$$

Also by applying Lemma 2.1 on (27), the desired result in (24) is obtained and this completes the proof of Theorem 2.3. With various choices of the parameters involved the following consequences are immediate.

Suppose that $\lambda=\mu=1$ in Theorem 2.3, then the following corollary follows:

Corollary 2.4 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{1,k}(\beta,1,A,B)$. Then

$$|a_{m+1}| \le \sqrt{\frac{8\alpha\delta^2(\alpha+2Bk)}{2\alpha(1+k)(1+m)\big((1+m)(B+1)[\alpha(1+\delta)(B+1)-2\delta(B-1)]+\delta(1+2m)[\delta(B-1)-\alpha(B+1)]\big)}}$$

and

$$|a_{2m+1}| \le \frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{\delta(\alpha+2Bk)}{(1+k)(1+m)[\delta(B-1)-\alpha(B+1)]}+\frac{1}{(1+2m)}\right]$$

where α and δ are as earlier defined.

Let $\lambda=1$ and $\mu=0$ in Theorem 2.3, then the following corollary is immediate.

Corollary 2.5 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{0,k}(\beta,1,A,B)$. Then

$$|a_{m+1}| \le \sqrt{\frac{8\alpha\delta^2(\alpha+2Bk)}{2\alpha(1+k)m\left(m(B+1)[\alpha(1+\delta)(B+1)-2\delta(B-1)]+\delta\begin{pmatrix}2(m+\frac{1}{2})[\delta(B-1)-\alpha(B+1)]\\ -\delta[B-1-\alpha(B+1)]\end{pmatrix}\right)}}$$

and

$$|a_{2m+1}| \le \frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{\delta(m+1)(\alpha+2Bk)}{(1+k)m^2[\delta(B-1)-\alpha(B+1)]}+\frac{1}{2m}\right]$$

where α and δ are as earlier defined.

Also if $A=1, B=-1$ and $k=0$ in Theorem 2.3, then we obtain the following corollary.

Corollary 2.6 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{\mu,0}(\beta,\lambda,1,-1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(\mu+m)(\mu+2m\lambda)}} \quad \text{and} \quad |a_{2m+1}| \leq 2(1-\beta)\left[\frac{(1-\beta)(m+1)}{(\mu+m\lambda)^2}+\frac{1}{\mu+2m\lambda}\right]$$

where α and δ are as earlier defined.

Suppose that we let $A=\lambda=1, B=-1$ and $k=0$ in Theorem 2.3, then corollary 2.7 follows.

Corollary 2.7 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{\mu,0}(\beta,1,1,-1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(\mu+m)(\mu+2m)}} \quad \text{and} \quad |a_{2m+1}| \leq 2(1-\beta)\left[\frac{(1-\beta)(m+1)}{(\mu+m)^2}+\frac{1}{\mu+2m}\right]$$

where α and δ are as earlier defined.

If $A=\lambda=1, B=-1$ and $k=\mu=0$ in Theorem 2.3, then corollary 2.8 follows.

Corollary 2.8 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{0,0}(\beta,1,1,-1)$. Then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m} \quad \text{and} \quad |a_{2m+1}| \leq \frac{2(1-\beta)}{m}\left[\frac{(1-\beta)(m+1)}{m}+\frac{1}{2}\right]$$

where α and δ are as earlier defined.

Suppose that we let $A=\lambda=\mu=1, B=-1$ and $k=0$ in Theorem 2.3, then corollary 2.9 follows.

Corollary 2.9 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{1,0}(\beta,1,1,-1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+m)(1+2m)}} \quad \text{and} \quad |a_{2m+1}| \leq 2(1-\beta)\left[\frac{(1-\beta)(m+1)}{(1+m)^2}+\frac{1}{1+2m}\right]$$

where α and δ are as earlier defined.

Suppose that $\lambda=\mu=1$ in Theorem 2.3, then the following corollary follows:

Corollary 2.10 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,m}^{1,k}(\beta,1,A,B)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{8\alpha\delta^2(\alpha+2Bk)}{4\alpha(1+k)[2(B+1)[\alpha(1+\delta)(B+1)-2\delta(B-1)]+3\delta[\delta(B-1)-\alpha(B+1)]]}}$$

and

$$|a_{2m+1}| \leq \frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{\delta(\alpha+2Bk)}{2(1+k)[\delta(B-1)-\alpha(B+1)]}+\frac{1}{3}\right].$$

Letting $\lambda=m=1$ and $\mu=0$ in Theorem 2.3, then corollary 2.11 follows.

Corollary 2.11 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,1}^{0,k}(\beta,1,A,B)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{8\alpha\delta^2(\alpha+2Bk)}{2\alpha(1+k)[(B+1)[\alpha(1+\delta)(B+1)-2\delta(B-1)]+\delta(3[\delta(B-1)-\alpha(B+1)]-\delta[B-1-\alpha(B+1)])]}}$$

and

$$|a_{2m+1}| \leq \frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}\left[\frac{2\delta(\alpha+2Bk)}{(1+k)[\delta(B-1)-\alpha(B+1)]}+\frac{1}{2}\right].$$

Letting $A=\lambda=\mu=m=1, B=-1$ and $k=0$ in Theorem 2.3, then corollary 2.12 follows:

Corollary 2.12 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,1}^{1,0}(\beta,1,1,-1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_{2m+1}| \leq \frac{1-\beta}{3}[5-3\beta].$$

If we let $A=\lambda=\mu=m=1, B=-1$ and $k=\beta=0$ in Theorem 2.3, then corollary 2.13 follows:

Corollary 2.13 Let $f(z)$ be the function of the form (6), if $f(z) \in M_{\Sigma,1}^{1,0}(\beta,1,1,-1)$. Then

$$|a_{m+1}| \leq \frac{2}{3} \quad \text{and} \quad |a_{2m+1}| \leq \frac{5}{3}.$$

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Conflict of Interest

The authors hereby declare that there is no conflict of interest in the present work.

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